Topic 5 - Exact Equations

Topic 5 – First order exact equations

Suppose you have a first order equation of the form:

$$\underbrace{M(x,y)} + \underbrace{N(x,y)} \cdot y' = 0$$

expressions with x and y

<u>Ex:</u>

$$\underbrace{2xy}_{M(x,y)} + \underbrace{(x^2-1)}_{N(x,y)}\, y' = 0$$

Suppose also that there exists a function $f(x,y)$ where

$$\frac{\partial f}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial f}{\partial y} = N(x,y)$$

Ex: $\underbrace{2xy}_{M(x,y)} + \underbrace{(x^2-1)}_{N(x,y)} y' = 0$

Let $f(x,y) = x^2 y - y$

Then,

$$\frac{\partial f}{\partial x} = 2xy + 0 = 2xy = M(x,y)$$

$$\frac{\partial f}{\partial y} = x^2 - 1 = N(x,y)$$

Suppose $\frac{\partial f}{\partial x} = M(x,y)$, $\frac{\partial f}{\partial y} = N(x,y)$.

Then,

$$M(x,y) + N(x,y) \cdot y' = 0$$

becomes

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0$$

Math 2130

$f(x,y)$ is a function of x and y

$y = y(x)$ is a function of x

chain rule:

$$\frac{df}{dx} = \frac{\partial f}{\partial x} \cdot \frac{d}{dx}(x) + \frac{\partial f}{\partial y} \cdot \frac{d}{dx}(y)$$

$$= \frac{\partial f}{\partial x}(1) + \frac{\partial f}{\partial y} \frac{dy}{dx}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}$$

Hence, $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0$

becomes $\frac{df}{dx} = 0$

So, for example the family of curves given by $f(x,y) = c$ where c is a constant will satisfy $\frac{df}{dx} = 0$.

<u>Summary</u>: If $\frac{\partial f}{\partial x} = M(x,y)$ and $\frac{\partial f}{\partial y} = N(x,y)$, then the equation $f(x,y) = c$ where c is any constant will give an implicit solution to

$$M(x,y) + N(x,y) \cdot y' = 0$$

If such an f exists then we say that

$$M(x,y) + N(x,y)\cdot y' = 0$$

is an <u>exact</u> equation

<u>Ex:</u> Consider

$$\underbrace{2xy}_{M(x,y)} + \underbrace{(x^2-1)}_{N(x,y)}y' = 0$$

Let

$$f(x,y) = x^2y - y$$

We have

$$\frac{\partial f}{\partial x} = 2xy = M(x,y)$$

$$\frac{\partial f}{\partial y} = x^2 - 1 = N(x,y)$$

Thus, $x^2y - y = c$ ← $f(x,y) = c$

where c is any constant is an implicit solution to $2xy + (x^2-1)y' = 0$

Check #1

Suppose $x^2y - y = c$.

Differentiate both sides with respect to x to get:

$$2xy + x^2\underbrace{y'}_{\frac{dy}{dx}} - \underbrace{y'}_{\frac{dy}{dx}} = 0$$

So,

$$2xy + (x^2-1)y' = 0$$

original equation

$\boxed{\text{Check \#2}}$

We can actually solve for y in our solution $x^2y - y = c$.

We get

$$y = \frac{c}{x^2-1}$$

Let's check if it solves the equation. We have

$$y = c(x^2-1)^{-1}$$

$$y' = -c(x^2-1)^{-2} \cdot (2x)$$

$$= \frac{-2cx}{(x^2-1)^2}$$

Plug this into the ODE to get

$$2xy + (x^2-1)y'$$

$$= 2x\underbrace{\left(\frac{c}{x^2-1}\right)}_{y} + (x^2-1)\underbrace{\left(\frac{-2cx}{(x^2-1)^2}\right)}_{y'}$$

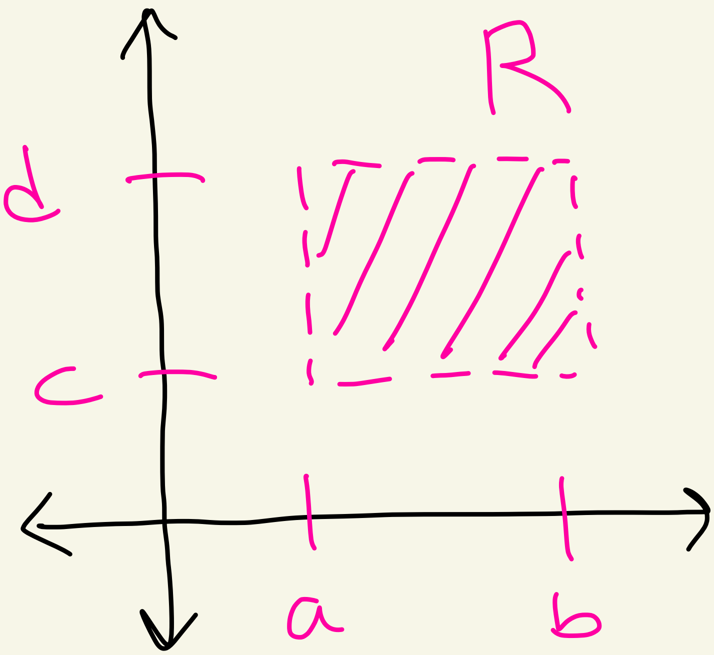
$$= \frac{2xc}{x^2-1} + \frac{-2cx}{x^2-1} = 0$$

Thus, $y = \frac{c}{x^2-1}$ solves

$$2xy + (x^2-1)y' = 0$$

When does such an f exist?

Theorem Let $M(x,y)$ and $N(x,y)$ be continuous and have continuous first partial derivatives in some rectangle R defined by $a<x<b$ and $c<x<d$.



Then,

$M(x,y)+N(x,y)\,y'=0$

will be exact if and only if

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Here a,b,c,d can be $\pm\infty$

Proof: See the end of these notes if interested $\boxtimes$

Ex: Consider the previous equation

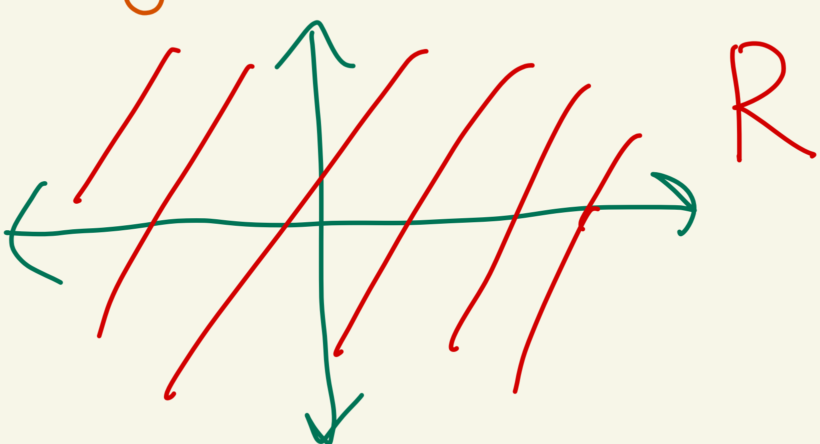
$$\underbrace{2xy}_{M(x,y)} + \underbrace{(x^2-1)}_{N(x,y)} y' = 0$$

We have

$$\left.\begin{aligned} M(x,y) &= 2xy \\ N(x,y) &= x^2-1 \end{aligned}\right\} \text{these are continuous everywhere}$$

And

$$\left.\begin{aligned} \frac{\partial M}{\partial x} &= 2y & \frac{\partial N}{\partial x} &= 2x \\ \frac{\partial M}{\partial y} &= 2x & \frac{\partial N}{\partial y} &= 0 \end{aligned}\right\} \text{these are continuous everywhere}$$



R is the entire xy-plane

And,

$$\frac{\partial M}{\partial y} = 2x$$

Equal!

$$\frac{\partial N}{\partial x} = 2x$$

Thus, $2xy + (x^2-1)y' = 0$ is exact, that is there exists $f(x,y)$ where

$$\frac{\partial f}{\partial x} = M(x,y) \text{ and } \frac{\partial f}{\partial y} = N(x,y)$$

Ex: Above we saw that

$$\underbrace{2xy}_{M(x,y)} + \underbrace{(x^2-1)}_{N(x,y)} y' = 0$$

is exact. We saw that a solution is given by $x^2y - y = c$ where $f(x,y) = x^2y - y$.

How did I find such an f?
Pretend like we don't know f.
We need f where

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= 2xy \quad (1) \\ \frac{\partial f}{\partial y} &= x^2 - 1 \quad (2) \end{aligned}}$$

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= M(x,y) \\ \frac{\partial f}{\partial y} &= N(x,y) \end{aligned}}$$

Integrate ① with respect to x to get

$$\boxed{f(x,y) = x^2y + \underbrace{C(y)}_{\text{constant with respect to } x} \qquad (*)}$$

Integrate ② with respect to y to get

$$\boxed{f(x,y) = x^2y - y + \underbrace{D(x)}_{\text{constant with respect to } y} \qquad (**)}$$

Now set $(*)$ and $(**)$ equal since they both equal $f(x,y)$ to get:

$$x^2y + C(y) = x^2y - y + D(x)$$

Now cancel out the common terms to get

$$C(y) = -y + \underbrace{D(x)}_{0}$$

To make these equal set $C(y) = -y$ and $D(x) = 0$.
Plug either into $(*)$ or $(**)$ to find f.
Plugging $C(y) = -y$ into $(*)$ gives

$$f(x,y) = x^2y - y$$

Answer: $x^2y - y = c$ solves $2xy + (x^2-1)y' = 0$ where c is any constant. Like we had above.

Ex:

Consider the initial value problem

$$(e^x + y) + (2 + x + ye^y)y' = 0$$
$$y(0) = 1$$

Let's solve

$$\underbrace{(e^x + y)}_{M(x,y)} + \underbrace{(2 + x + ye^y)}_{N(x,y)} y' = 0$$

But first is it exact so we can use our method?

We have

$$M(x,y) = e^x + y$$
$$N(x,y) = 2 + x + ye^y$$

these are continuous everywhere

$$\frac{\partial M}{\partial x} = e^x \qquad \frac{\partial N}{\partial x} = 1$$

$$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial y} = 1 \cdot e^y + y \cdot e^y$$

<u>Check:</u> $\frac{\partial N}{\partial x} = 1 = \frac{\partial M}{\partial y}$

So, the ODE equation is exact.

Thus, the equation

$$(e^x + y) + (2 + x + ye^y)\, y' = 0$$

is exact. So there must exist

$f(x,y)$ that satisfies:

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= e^x + y \quad (1) \\ \frac{\partial f}{\partial y} &= 2 + x + ye^y \quad (2) \end{aligned}}$$

$\frac{\partial f}{\partial x} = M(x,y)$

$\frac{\partial f}{\partial y} = N(x,y)$

Integrate (1) with respect to x to get

$$\boxed{f(x,y) = e^x + yx + \underbrace{C(y)}_{\text{constant with respect to } x}} \quad (*)$$

Now integrate (2) with respect to y to get

$$\boxed{f(x,y) = 2y + xy + ye^y - e^y + \underbrace{D(x)}_{\text{constant with respect to } y}} \quad (**)$$

$$\int ye^y\,dy = ye^y - \int e^y\,dy = ye^y - e^y$$

$u = y \quad dv = e^y dy$

$du = dy \quad v = e^y$

Set (*) equal to (**) since they both equal $f(x,y)$ to get:

$$e^x + yx + C(y) = 2y + xy + ye^y - e^y + D(x)$$

Cancel out common terms to get

$$e^x + C(y) = 2y + ye^y - e^y + D(x)$$

Set $C(y) = 2y + ye^y - e^y$ and $D(x) = e^x$.

Plug either into (*) or (**) to find $f(x,y)$.

Plugging $C(y)$ into (*) gives

$$f(x,y) = e^x + yx + 2y + ye^y - e^y$$

A solution to

$$(e^x+y)+(2+x+ye^y)y'=0$$

is

$$e^x+yx+2y+ye^y-e^y=c$$

where c is a constant.

$f(x,y)=c$

Now let's find a solution where $y(0)=1$.

Plug $x=0$, $y=1$ into our solution.

We get:

$$\underbrace{e^0}_{1}+\underbrace{(1)(0)}_{0}+\underbrace{2(1)}_{2}+\underbrace{(1)e^1-e^1}_{0}=c$$

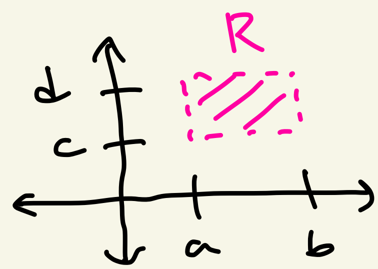
So $c=3$. And an answer to the initial-value problem is $e^x+yx+2y+ye^y-e^y=3$

(this is an implicit solution since you can't solve for y)

Below I put a proof of the main theorem in this topic. It's mainly for me. But if you're interested, see below.

Let's prove this theorem.

Theorem: Let $M(x,y)$ and $N(x,y)$ be continuous and have continuous first partial derivatives in some rectangle R defined by $a<x<b$ and $c<y<d$



Then

$$M(x,y)+N(x,y)\cdot y'=0$$

is exact if and only if

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

proof: For simplicity suppose R is the entire xy-plane and that M and N are continuous for all (x,y) and so are their partial derivatives.

($\Rightarrow$) First suppose that $M+Ny'=0$ is exact. Then there exists f where $\frac{\partial f}{\partial x}=M$ and $\frac{\partial f}{\partial y}=N$.

Then, $\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}f\right)=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}f\right)=\frac{\partial N}{\partial x}$.

Calc III - Clairauts thm applied to M_y and N_x

(⇐) Suppose now that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. We will show that this implies that $M + Ny' = 0$ is exact.

Since M is continuous we can define

$$f(x,y) = \int M(x,y)\,dx + g(y) \qquad (*)$$

where g is any function of y.

Here we get that $\boxed{\frac{\partial f}{\partial x} = M.}$

We want to now find $g(y)$ where

$$N = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)\,dx + g'(y)$$

We will need

$$g'(y) = N - \frac{\partial}{\partial y}\int M(x,y)\,dx$$

To do this we can show that the RHS is just a function of y and hence we can integrate it with respect to y to get $g(y)$.

We have that

$$\frac{\partial}{\partial x}\left(N - \frac{\partial}{\partial y}\int M(x,y)\,dx\right) =$$

$$= \frac{\partial N}{\partial x} - \frac{\partial}{\partial x}\frac{\partial}{\partial y}\int M(x,y)\,dx$$

$$= \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\frac{\partial}{\partial x}\int M(x,y)\,dx$$

$$= \frac{\partial N}{\partial x} - \frac{\partial}{\partial y} M$$

$$= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

$$= 0$$

since $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

Thus, such a $g(y)$ exists.

And

$$f(x,y) = \int M(x,y)\,dx + \int \left(N(x,y) - \int M(x,y)\,dx \right) dy$$

will satisfy $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$.

$\blacksquare$